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Algebraic frames in Priestley duality

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Abstract. We characterize Priestley spaces of algebraic, arithmetic, coherent, and Stone frames. As a corollary, we derive the well-known dual equivalences in pointfree topology involving various categories of algebraic frames.

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Keywords. Pointfree topology, Priestley duality, Algebraic frame, Coherent frame, Stone frame, Spectral space, Stone space.

1. Introduction

A complete lattice is algebraic provided every element is a join of compact elements. Algebraic lattices arise naturally in different contexts. For example, the lattice of congruences of any algebra is algebraic, and up to isomorphism, every algebraic lattice arises this way (see, e.g., [15]). It is a well-known result of Nachbin [30] (see also [14]) that algebraic lattices are exactly the ideal lattices of join-semilattices. If an algebraic lattice L is distributive, then the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$ holds, and hence L is a frame. Such frames are known as algebraic frames and have been the subject of study in pointfree topology and domain theory (see, e.g., [20, 31]).

There is a well-developed duality theory for the category AlgFrm of algebraic frames and its various subcategories such as the categories of arithmetic frames (also known as M-frames), coherent frames, and Stone frames. Indeed, a frame L is algebraic iff it is the frame of opens of a compactly based sober space X [20, p. 423]. In addition, L is arithmetic iff X is stably compactly based, L is coherent iff X is spectral, and L is a Stone frame iff X is a Stone space (see Section 2 for details).

The duality theory for algebraic frames is a restriction of the well-known Hofmann-Lawson duality [24]. We recall (see, e.g., [31, p. 135]) that a frame L is *continuous* if the way-below relation $\ll$ is approximating. In addition, L

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$$\begin{array}{ccccccc} \mathsf{ConFrm} & \geqslant & \mathsf{StCFrm} & \geqslant & \mathsf{StKFrm} & \geqslant & \mathsf{KRFrm} \\ \veeeq & & \veeeq & & \veeeq & & \veeeq \\ \mathsf{AlgFrm} & \geqslant & \mathsf{AriFrm} & \geqslant & \mathsf{CohFrm} & \geqslant & \mathsf{StoneFrm} \end{array}$$

Figure 1. Inclusion relationships between categories of continuous and algebraic frames

is *stably continuous* if $\ll$ is stable ($a \ll b, c$ implies $a \ll b \wedge c$) and L is *stably compact* if moreover L is compact. We also recall (see, e.g., [31, p. 89]) that L is *regular* if the well-inside relation $\prec$ is approximating. A regular frame L is *compact regular* if it is furthermore compact. Since in compact regular frames the way-below and well-inside relations coincide, every compact regular frame is stably compact. Figure 1 describes the correspondence between various categories of continuous and algebraic frames, where the categories are defined in Tables 1 and 2 and $\leqslant$ stands for "is a full subcategory of."

By the well-known Priestley duality [32,33], the category of bounded distributive lattices is dually equivalent to the category of Priestley spaces. Pultr and Sichler [34] provided a restricted version of Priestley duality for the category Frm of frames and frame homomorphisms. This line of research was further developed by several authors (see, e.g., [35,10,8,9,1,2]). In [12], we obtained Priestley duality for ConFrm and its subcategories listed in the first row of Figure 1. The resulting (dual) equivalences are outlined in Figure 5. The aim of this paper is to further study Priestley duality for AlgFrm and its subcategories listed in the second row of Figure 1. This requires characterizing Priestley spaces of algebraic, arithmetic, coherent, and Stone frames.

The paper is organized as follows. In Section 2, we describe the above categories of continuous and algebraic frames, as well as the corresponding

Table 1. Categories of continuous frames

Category	Objects	Morphisms
ConFrm	Continuous frames	Proper frame homomorphisms
StCFrm	Stably continuous frames	Proper frame homomorphisms
StKFrm	Stably compact frames	Proper frame homomorphisms
KRFrm	Compact regular frames	Frame homomorphisms

Table 2. Categories of algebraic frames

Category	Objects	Morphisms
AlgFrm	Algebraic frames	Coherent frame homomorphisms
AriFrm	Arithmetic frames	Coherent frame homomorphisms
CohFrm	Coherent frames	Coherent frame homomorphisms
StoneFrm	Stone frames	Frame homomorphisms

categories of locally compact and compactly based sober spaces. Section 3 recalls Priestley duality for various categories of continuous frames. In Section 4, we characterize Priestley spaces of algebraic frames. Consequently, we obtain a new proof of the duality between AlgFrm and KBSob. Finally, in Section 5, we characterize Priestley spaces of arithmetic, coherent, and Stone frames. In each case, this yields a new proof of the duality between the corresponding categories of algebraic frames and compactly based spaces. We conclude the paper by connecting Priestley spaces of coherent frames and Stone frames to Priestley duality for bounded distributive lattices and Stone duality for boolean algebras.

2. Continuous and algebraic frames

A *frame* is a complete lattice L satisfying the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for every $a \in L$ and $S \subseteq L$. A *frame homomorphism* is a map between frames that preserves finite meets and arbitrary joins. Let Frm be the category of frames and frame homomorphisms. A frame is *spatial* if completely prime filters separate elements of L. Let SFrm be the full subcategory of Frm consisting of spatial frames.

As usual, we write $\ll$ for the *way-below* relation in a frame L and recall that $a \ll b$ provided for each $S \subseteq L$ we have $b \le \bigvee S$ implies $a \le \bigvee T$ for some finite $T \subseteq S$. We call $a \in L$ *compact* if $a \ll a$ and L *compact* if its top element is compact. We write $K(L)$ for the collection of compact elements of L.

We also recall that the *well-inside* relation on L is defined by $a \prec b$ if $a^* \vee b = 1$, where $a^* := \bigvee\{x \in L \mid a \wedge x = 0\}$ is the pseudocomplement of a. An element $a \in L$ is *complemented* if $a \prec a$. Let $C(L)$ be the collection of complemented elements of L. It is well known that if L is compact, then $a \prec b$ implies $a \ll b$; and if L is regular, then $a \ll b$ implies $a \prec b$. Thus, in compact regular frames, the two relations $\ll$ and $\prec$ coincide, and hence $K(L) = C(L)$.

A frame homomorphism $h \colon L \to M$ between continuous frames is *proper* if it preserves $\ll$; that is, $a \ll b$ implies $h(a) \ll h(b)$ for all $a, b \in L$. Let ConFrm be the category of continuous frames and proper frame homomorphisms. We write StCFrm and StKFrm for the full subcategories of ConFrm consisting of stably continuous and stably compact frames, respectively. We also let KRFrm be the full subcategory of Frm consisting of compact regular frames. Since every frame homomorphism between compact regular frames is proper, KRFrm is a full subcategory of StKFrm.

Definition 2.1.

(1) ([31, p. 142]) A frame L is *algebraic* if $a = \bigvee\{b \in K(L) \mid b \le a\}$ for all $a \in L$.

(2) ([29, p. 64]) A frame homomorphism $h \colon L \to M$ is *coherent* if $a \in K(L)$ implies $h(a) \in K(M)$.

(3) Let AlgFrm be the category of algebraic frames and coherent frame homomorphisms.

Remark 2.2. It is easy to see that every algebraic frame is continuous, and that a frame homomorphism between coherent frames is coherent iff it is proper. Consequently, AlgFrm is a full subcategory of ConFrm.

Definition 2.3.
(1) A frame L is *arithmetic* if it is algebraic and $\ll$ is stable.
(2) Let AriFrm be the full subcategory of AlgFrm consisting of arithmetic frames.

Remark 2.4.
(1) In [20] a lattice is called arithmetic if the binary meet of compact elements is compact. For algebraic lattices this is equivalent to $\ll$ being stable (see, e.g. [20, Proposition I$-$4.8]).
(2) Arithmetic frames are also called M-frames; see, e.g., [25,13].

Definition 2.5.
(1) ([29, p. 63–64]) A frame L is *coherent* if L is arithmetic and compact.
(2) Let CohFrm be the full subcategory of AriFrm consisting of coherent frames.

The next definition is well known (see, e.g., [29,5,27]). We thank Joanne Walters-Wayland for pointing out to us that the terminology of Stone frames originated from Banaschewski's University of Cape Town lecture notes (1988).

Definition 2.6.
(1) A frame L is *zero-dimensional* if $a = \bigvee\{b \in C(L) \mid b \leq a\}$ for all $a \in L$.
(2) A *Stone frame* is a compact zero-dimensional frame.
(3) Let $\mathsf{StoneFrm}$ be the full subcategory of Frm consisting of Stone frames.

Remark 2.7. Clearly $\mathsf{StoneFrm}$ is a full subcategory of KRFrm. Moreover, since every frame homomorphism preserves $\prec$ and in Stone frames $\prec$ coincides with $\ll$, we have that $\mathsf{StoneFrm}$ is a full subcategory of CohFrm.

The categories of algebraic and continuous frames relate to each other as shown in Figure 1. We next turn our attention to the corresponding categories of topological spaces. The following definitions are well known (see, e.g., [20, pp. 43–44]). A closed subset of a topological space X is *irreducible* if it cannot be written as the union of two proper closed subsets. We call X *sober* if each irreducible subset is the closure of a unique point in X, and *locally compact* if for every open set U and $x \in U$ there are an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$.

In view of [20, Lemma VI$-$6.21], we call a continuous map $f\colon X \to Y$ between locally compact sober spaces *proper* if $f^{-1}(K)$ is compact for each compact saturated set $K \subseteq Y$. Let LKSob be the category of locally compact sober spaces and proper maps between them.

A topological space X is *coherent* if the intersection of two compact saturated sets is compact ([20, p. 474]), and X is *stably locally compact* if it

Table 3. Categories of locally compact sober spaces

Category	Objects	Morphisms
LKSob	Locally compact sober spaces	Proper maps
StLKSp	Stably locally compact spaces	Proper maps
StKSp	Stably compact spaces	Proper maps
KHaus	Compact Hausdorff spaces	Continuous maps

is locally compact, sober, and coherent. Let StLKSp be the full subcategory of LKSob consisting of stably locally compact spaces.

A compact stably locally compact space is a *stably compact* space ([20, p. 476]). We write StKSp for the full subcategory of StLKSp consisting of stably compact spaces. Also, we denote by KHaus the category of compact Hausdorff spaces and continuous maps. Since a continuous map between compact Hausdorff spaces is proper, KHaus is a full subcategory of StKSp. Table 3 lists the above categories of locally compact sober spaces.

We now shift our focus to compactly based spaces. We recall that a continuous map $f: X \to Y$ is *coherent* if $f^{-1}(U)$ is compact for each compact open $U \subseteq Y$.

Definition 2.8.

(1) ([16, p. 2063]) A topological space X is *compactly based* if it has a basis of compact open sets. Let KBSob be the category of compactly based sober spaces and coherent maps.

(2) A compactly based space X is *stably compactly based* if it is sober and the intersection of two compact opens is compact. Let StKBSp be the full subcategory of KBSob consisting of stably compactly based spaces.

(3) ([22, p. 43]) A stably compactly based space X is a *spectral* space if it is compact. Let Spec be the full subcategory of StKBSp consisting of spectral spaces.

(4) ([29, p. 70]) A *Stone* space is a zero-dimensional compact Hausdorff space. Let Stone be the category of Stone spaces and continuous maps.

Table 4 lists the categories of compactly based sober spaces of Definition 2.8.

Remark 2.9. It is easy to see that Stone is a full subcategory of Spec (see, e.g., [29, p. 71]). To see that KBSob is a full subcategory of LKSob, it is sufficient to observe that a continuous map between compactly based sober spaces is coherent iff it is proper. For this it is enough to observe that in a compactly based space X, every compact saturated set is an intersection of compact opens. To see this, let $K \subseteq X$ be compact saturated. It suffices to show that for each $x \notin K$ there is a compact open U containing K and missing x. For each $y \in K$ there is a compact open U_y such that $y \in U_y$ and $x \notin U_y$. Therefore, $K \subseteq \bigcup\{U_y \mid y \in K\}$. By compactness of K and the fact that a finite union of compact sets is compact, there is a compact open U such that $K \subseteq U$ and $x \notin U$.

Table 4. Categories of compactly based sober spaces

Category	Objects	Morphisms
KBSob	Compactly based sober spaces	Coherent maps
StKBSp	Stably compactly based spaces	Coherent maps
Spec	Spectral spaces	Coherent maps
Stone	Stone spaces	Continuous maps

The diagram in Figure 2 gives the correspondence between various categories of locally compact and compactly based sober spaces.

There is a well-known dual adjunction between the categories Top and Frm, which restricts to a dual equivalence between the categories Sob and SFrm (see, e.g., [29, Section II−1]). Further restrictions of this equivalence yield the following well-known duality results for continuous frames:

Theorem 2.10.
(1) ConFrm *is dually equivalent to* LKSob.
(2) StCFrm *is dually equivalent to* StLKSp.
(3) StKFrm *is dually equivalent to* StKSp.
(4) KRFrm *is dually equivalent to* KHaus.

We thus arrive at the diagram in Figure 3, where $\leftrightsquigarrow$ represents dual equivalence.

Remark 2.11. Theorem 2.10(1) is known as Hofmann-Lawson duality [24] (see also [20, Proposition V−5.20]). The origins of Theorems 2.10(2) and 2.10(3) can be traced back to [21, 28, 36, 4] (see also [20, Section VI−7.4]). Finally, Theorem 2.10(4) is known as Isbell duality [26] (see also [6] or [29, Section VII−4]).

We next describe the duality results for algebraic frames. One of the earliest references is probably [23, Theorem 5.7] (see also [20, p. 423]), where the

$$\begin{array}{ccccccc} \mathsf{LKSob} & \geqslant & \mathsf{StLKSp} & \geqslant & \mathsf{StKSp} & \geqslant & \mathsf{KHaus} \\ \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} \\ \mathsf{KBSob} & \geqslant & \mathsf{StKBSp} & \geqslant & \mathsf{Spec} & \geqslant & \mathsf{Stone} \end{array}$$

Figure 2. Inclusion relationships between categories of locally compact and compactly based sober spaces

$$\begin{array}{ccccccc} \mathsf{ConFrm} & \geqslant & \mathsf{StCFrm} & \geqslant & \mathsf{StKFrm} & \geqslant & \mathsf{KRFrm} \\ \updownsquigarrow & & \updownsquigarrow & & \updownsquigarrow & & \updownsquigarrow \\ \mathsf{LKSob} & \geqslant & \mathsf{StLKSp} & \geqslant & \mathsf{StKSp} & \geqslant & \mathsf{KHaus} \end{array}$$

Figure 3. Correspondence between categories of continuous frames and locally compact spaces

$$\begin{array}{ccccccc}
\mathsf{AlgFrm} & \geqslant & \mathsf{AriFrm} & \geqslant & \mathsf{CohFrm} & \geqslant & \mathsf{StoneFrm} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathsf{KBSob} & \geqslant & \mathsf{StKBSp} & \geqslant & \mathsf{Spec} & \geqslant & \mathsf{Stone}
\end{array}$$

FIGURE 4. Correspondence between categories of algebraic frames and compactly based spaces

dualities for AlgFrm, AriFrm, and CohFrm are stated. The duality for CohFrm is also described in [3,4] and [29, Section II.3]. This further reduces to the duality for StoneFrm (see, e.g., [5] or [27, Chapter IV]).

Theorem 2.12.
(1) AlgFrm *is dually equivalent to* KBSob.
(2) AriFrm *is dually equivalent to* StKBSp.
(3) CohFrm *is dually equivalent to* Spec.
(4) StoneFrm *is dually equivalent to* Stone.

We thus arrive at the diagram in Figure 4.

Remark 2.13. The proof of Theorem 2.12 can easily be deduced from Theorem 2.10 and the fact that AlgFrm and KBSob are full subcategories of ConFrm and LKSob, respectively. But it is easy to give a direct proof of Theorem 2.12 which does not rely on Theorem 2.10. For this it is sufficient to observe that every algebraic frame is spatial. Let L be an algebraic frame. Then Scott-open filters separate elements of L. To see this, if $a \not\leq b$, then there is $k \in K(L)$ such that $k \leq a$ but $k \not\leq b$. Thus, $\uparrow k$ is a Scott-open filter containing a and missing b. It is left to observe that the Prime Ideal Theorem implies that L is spatial iff Scott-open filters separate elements of L (see, e.g., [17, p. 265] or [11, Corollary 5.9(2)]).

3. Priestley duality for continuous frames

As we pointed out in the introduction, Pultr and Sichler [34] restricted Priestley duality for bounded distributive lattices to the category of frames. In this section we briefly recall Pultr-Sichler duality and its restriction to various categories of continuous frames.

Definition 3.1.
(1) A *Priestley space* is a Stone space X with a partial order $\leq$ such that clopen upsets separate points.
(2) An *L-space* (*localic space*) is a Priestley space such that $\operatorname{cl} U$ is an open upset for each open upset U of X.
(3) An *L-morphism* is a continuous order-preserving map $f\colon X \to Y$ between L-spaces such that $f^{-1} \operatorname{cl} U = \operatorname{cl} f^{-1} U$ for every open upset U of Y.
(4) Let LPries be the category of L-spaces and L-morphisms.

Theorem 3.2 (Pultr-Sichler [34, Corollary 2.5]). Frm *is dually equivalent to* LPries.

Remark 3.3. The functors $\mathscr{X}\colon \mathsf{Frm} \to \mathsf{LPries}$ and $\mathscr{L}\colon \mathsf{LPries} \to \mathsf{Frm}$ establishing Pultr-Sichler duality are the restrictions of the functors establishing Priestley duality. We recall that the *Priestley space* of a frame L is the set X_L of prime filters of L ordered by inclusion and topologized by the subbasis $\{\varphi(a) \mid a \in L\} \cup \{\varphi(a)^c \mid a \in L\}$, where φ is the Stone map given by $\varphi(a) = \{x \in X_L \mid a \in x\}$ for each $a \in L$. The functor $\mathscr{X}$ sends a frame L to its Priestley space X_L and a frame homomorphism $h\colon L \to M$ to the L-morphism $h^{-1}\colon X_M \to X_L$. The functor $\mathscr{L}$ sends an L-space X to the frame $\mathrm{ClopUp}(X)$ of clopen upsets of X and an L-morphism $f\colon X \to Y$ to the frame homomorphism $f^{-1}\colon \mathrm{ClopUp}(Y) \to \mathrm{ClopUp}(X)$.

Remark 3.4. Since frames are complete Heying algebras (see, e.g., [31, p. 332]), there is a close connection between Pultr-Sichler duality and Esakia duality for Heyting algebras [18]. We recall that an *Esakia space* is a Priestley space with the additional property that the partial order $\leq$ is continuous (the downset of each clopen is clopen). A Priestley space is an Esakia space iff the closure $\mathrm{cl}\, U$ of each open upset U is an upset (see, e.g, [8, Lemma 4.2]). Thus, L-spaces are those Esakia spaces in which $\mathrm{cl}\, U$ is not just an upset but an open upset. Such Esakia spaces are called *extremally order-disconnected* as they generalize extremally disconnected Stone spaces. Thus, a Priestley space is an L-space iff it is an extremally order-disconnected Esakia space.

We next characterize Priestley spaces of spatial frames.

Definition 3.5. Let X be an L-space.

(1) The set $Y := \{y \in X \mid {\downarrow} y \text{ is clopen}\}$ is called the *spatial part of* X.
(2) We call X an *SL-space* if Y is dense in X.
(3) Let $\mathsf{SLPries}$ be the full subcategory of LPries consisting of SL-spaces.

Let L be a frame. Recall (see, e.g., [31, p. 15]) that a *point* of L is a completely prime filter, and that the set $\mathrm{pt}(L)$ of points of L is topologized by $\{\varphi(a) \cap \mathrm{pt}(L) \mid a \in L\}$. We will refer to $\mathrm{pt}(L)$ as the *space of points of* L.

Remark 3.6. Let X be an L-space and Y the spatial part of X.

(1) We view Y as a topological space, where $V \subseteq Y$ is open iff $V = U \cap Y$ for some $U \in \mathrm{ClopUp}(X)$. If X is the Priestley space of a frame L, then the spatial part Y of X is exactly the space of points of L (see, e.g., [1, Lemma 5.3(1)]).
(2) If X is an SL-space, then $\mathrm{cl}(U \cap Y) = U$ for each $U \in \mathrm{ClopUp}(X)$. Therefore, the assignment $U \mapsto U \cap Y$ is an isomorphism from the poset of clopen upsets of X to the poset of open sets of Y. This will be utilized in what follows.

Theorem 3.7 ([12, Section 4]). $\mathsf{SLPries}$ *is equivalent to* Sob *and dually equivalent to* SFrm.

Remark 3.8.

(1) The dual equivalence between SFrm and $\mathsf{SLPries}$ is obtained by restricting the functors establishing Pultr-Sichler duality.

(2) The equivalence between SLPries and Sob is obtained as follows. Let $\mathscr{Y}: \mathsf{LPries} \to \mathsf{Sob}$ be the functor that sends an L-space X to to its spatial part Y, and an L-morphism $f: X_1 \to X_2$ to its restriction $g: Y_1 \to Y_2$. Then $\mathscr{Y}$ restricts to an equivalence between SLPries and Sob (see, e.g., [12, Corollary 4.19]).

(3) As an immediate consequence of Theorem 3.7, we obtain the well-known duality between SFrm and Sob.

We now turn our attention to Priestley spaces of continuous frames.

Definition 3.9. Let X be an L-space.

(1) For $U, V \in \mathrm{ClopUp}(X)$, define $V \ll U$ provided for each open upset W of X we have $U \subseteq \mathrm{cl}\, W$ implies $V \subseteq W$.

(2) For $U \in \mathrm{ClopUp}(X)$, define the *kernel of* U as

$$\ker U = \bigcup \{V \in \mathrm{ClopUp}(X) \mid V \ll U\}.$$

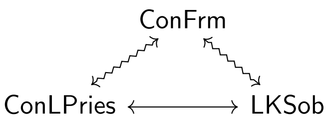
(3) We call X a *continuous L-space* provided $\ker U$ is dense in U for each $U \in \mathrm{ClopUp}(X)$.

(4) An L-morphism $f: X_1 \to X_2$ is *proper* if $f^{-1}(\ker U) \subseteq \ker f^{-1}(U)$ for all $U \in \mathrm{ClopUp}(X_2)$.

(5) Let ConLPries be the category of continuous L-spaces and proper L-morphisms.

Theorem 3.10 ([12, Section 5]). ConLPries *is equivalent to* LKSob *and dually equivalent to* ConFrm.

As a corollary, we obtain Hofmann-Lawson duality that ConFrm is dually equivalent to LKSob (see Theorem 2.10(1)). We thus arrive at the following diagram which commutes up to natural isomorphism, where $\leftrightarrow$ represents equivalence.



We next describe Priestley spaces of stably continuous and stably compact frames. For the next definition see [12, Section 6]. The notion of L-compact goes back to [34,35].

Definition 3.11.

(1) (a) An L-space X is *kernel-stable* if $\ker(U \cap V) = \ker U \cap \ker V$ for all $U, V \in \mathrm{ClopUp}(X)$,

(b) A *stably continuous L-space* is a kernel-stable continuous L-space.

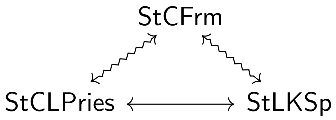
(c) Let StCLPries be the full subcategory of ConLPries consisting of stably continuous L-spaces.

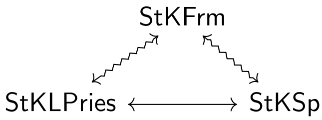
(2) (a) An L-space X is *L-compact* if $X = \ker X$.

(b) A *stably compact L-space* is an L-compact stably continuous L-space.
(c) Let StKLPries be the full subcategory of StCLPries consisting of stably compact L-spaces.

Theorem 3.12 ([12, Section 6]).
(1) StCLPries *is equivalent to* StLKSp *and dually equivalent to* StCFrm.
(2) StKLPries *is equivalent to* StKSp *and dually equivalent to* StKFrm.

As a consequence, we obtain the following well-known dualities for stably continuous frames: StCFrm is dually equivalent to StLKSp (see Theorem 2.10(2)) and StKFrm is dually equivalent to StKSp (see Theorem 2.10(3)).





We conclude this section by describing Priestley spaces of compact regular frames. The next definition appeared in [9, Section 3] and [12, Section 7]. The notion of regular L-space goes back to [34].

Definition 3.13. Let X be an L-space.
(1) For $U, V \in \mathrm{ClopUp}(X)$, define $V \prec U$ provided ${\downarrow}V \subseteq U$.
(2) For $U \in \mathrm{ClopUp}(X)$, define the *regular part of* U as

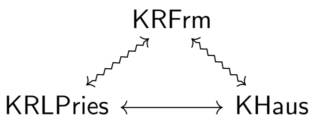
$$\operatorname{reg} U = \bigcup \{V \in \mathrm{ClopUp}(X) \mid V \prec U\}.$$

(3) We call X a *regular L-space* if $\operatorname{reg} U$ is dense in U for each $U \in \mathrm{ClopUp}(X)$.
(4) We call X a *compact regular L-space* if X is a regular L-space that is L-compact.
(5) Let KRLPries be the full subcategory of LPries consisting of compact regular L-spaces.

Remark 3.14. Every L-morphism between compact regular L-spaces is proper (see [12, Theorem 7.18(2)]), and every compact regular L-space is a stably compact L-space (see [12, Theorem 7.17]). Thus, KRLPries is a full subcategory of StKLPries.

Theorem 3.15 ([12, Section 7]). KRLPries *is equivalent to* KHaus *and dually equivalent to* KRFrm.

As a corollary, we obtain Isbell duality that KRFrm is dually equivalent to KHaus (see Theorem 2.10(4)).



$$
\begin{array}{ccccccc}
\mathsf{ConFrm} & \geqslant & \mathsf{StCFrm} & \geqslant & \mathsf{StKFrm} & \geqslant & \mathsf{KRFrm} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathsf{ConLPries} & \geqslant & \mathsf{StCLPries} & \geqslant & \mathsf{StKLPries} & \geqslant & \mathsf{KRLPries} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathsf{LKSob} & \geqslant & \mathsf{StLKSp} & \geqslant & \mathsf{StKSp} & \geqslant & \mathsf{KHaus}
\end{array}
$$

FIGURE 5. Equivalences and dual equivalences between categories of continuous frames, continuous L-spaces, and locally compact sober spaces

We thus have the diagram in Figure 5. The categories of continuous L-spaces are listed in Table 5.

In what follows, we will obtain a similar picture of equivalences and dual equivalences when the above categories of continuous frames are replaced by the corresponding full subcategories of algebraic frames.

4. Priestley duality for algebraic frames

In this section we describe algebraic frames in the language of Priestley spaces. We then connect the Priestley duals of algebraic frames with compactly based sober spaces to derive the well-known duality between AlgFrm and KBSob mentioned in Theorem 2.12(1).

Let X be an L-space and Y the spatial part of X. We recall (see [11, Definition 5.2]) that a closed upset F of X is a *Scott upset* if $F = {\uparrow}(F \cap Y)$. We have the following characterization of Scott upsets, where we write $\min F$ for the set of minimal points of F.

Lemma 4.1 ([11, Lemma 5.1]). *Let X be an L-space and let F be a closed upset of X.*

(1) *F is a Scott upset.*
(2) $\min F \subseteq Y$.
(3) *$F \subseteq \operatorname{cl} U \implies F \subseteq U$ for each open upset U of X.*

We denote by $\mathrm{ClopSUp}(X)$ the collection of all clopen Scott upsets of X.

Definition 4.2. Let X be an L-space.

TABLE 5. Categories of continuous L-spaces

Category	Objects	Morphisms
ConLPries	Continuous L-spaces	Proper L-morphisms
StCLPries	Stably continuous L-spaces	Proper L-morphisms
StKLPries	Stably compact L-spaces	Proper L-morphisms
KRLPries	Compact regular L-spaces	L-morphisms

(1) For $U \in \mathrm{ClopUp}(X)$, define the *core* of U as

$$\operatorname{core} U = \bigcup\{V \subseteq U \mid V \in \mathrm{ClopSUp}(X)\}.$$

(2) Call X an *algebraic L-space* provided $\operatorname{core} U$ is dense in U for every $U \in \mathrm{ClopUp}(X)$.

Lemma 4.3. *Let X be an L-space and $U, V \in \mathrm{ClopUp}(X)$.*

(1) $\operatorname{core} U \subseteq \ker U \subseteq U$.
(2) $U \subseteq V$ *implies* $\operatorname{core} U \subseteq \operatorname{core} V$.
(3) *If X is an algebraic L-space, then X is a continuous L-space.*
(4) *U is a Scott upset iff* $\operatorname{core} U = U$.

Proof. (1) Suppose $x \in \operatorname{core} U$. Then there is $V \in \mathrm{ClopSUp}(X)$ such that $x \in V \subseteq U$. Let W be an open upset such that $U \subseteq \operatorname{cl} W$. Then $V \subseteq \operatorname{cl} W$, so $V \subseteq W$ by Lemma 4.1. Hence, $V \ll U$. Therefore, $x \in \ker U$, and so $\operatorname{core} U \subseteq \ker U$. That $\ker U \subseteq U$ follows from [12, Lemma 5.2(1)].

(2) This is obvious from the definition of the core.

(3) Let $U \in \mathrm{ClopUp}(X)$. Since X is an algebraic L-space, $\operatorname{core} U$ is dense in U. Therefore, $\ker U$ is dense in U by (1). Thus, X is a continuous L-space.

(4) First suppose that U is a Scott upset. By (1), $\operatorname{core} U \subseteq U$. Since U is a Scott upset, $U \subseteq \operatorname{core} U$. Thus, $\operatorname{core} U = U$. Conversely, suppose that $U = \operatorname{core} U$. Since U is compact, there are clopen Scott upsets $V_1, \dots, V_n \subseteq U$ such that $U = V_1 \cup \dots \cup V_n$. Because a finite union of Scott upsets is a Scott upset, U is a Scott upset. □

We next connect algebraic frames with algebraic L-spaces. Let L be a frame, X_L its Priestley space, and $a \in L$. To simplify notation, we write $\operatorname{core}(a)$ for $\operatorname{core} \varphi(a)$ and $\ker(a)$ for $\ker \varphi(a)$.

Lemma 4.4 ([12, Lemma 6.10]). *Let L be a frame and X_L its Priestley space. For $a \in L$, the following are equivalent.*

(1) *a is compact.*
(2) $\ker(a) = \varphi(a)$.
(3) *$\varphi(a)$ is a Scott upset.*

In particular, L is compact iff X_L is L-compact iff $\min X_L \subseteq Y_L$.

Theorem 4.5. *Let L be a frame and X_L its Priestley space.*

(1) *For $a \in L$, we have $a = \bigvee\{b \in K(L) \mid b \le a\}$ iff $\operatorname{core}(a)$ is dense in $\varphi(a)$.*
(2) *L is an algebraic frame iff X_L is an algebraic L-space.*

Proof. (1) It is well known (see, e.g., [7, Lemma 2.3]) that

$$\varphi\left(\bigvee S\right) = \operatorname{cl}\left(\bigcup\{\varphi(s) \mid s \in S\}\right)$$

for each $S \subseteq L$. Therefore, by Lemma 4.4 we have $a = \bigvee\{b \in K(L) \mid b \le a\}$ iff

$$\varphi(a) = \operatorname{cl}\left(\bigcup\{\varphi(b) \in \mathrm{ClopSUp}(X_L) \mid \varphi(b) \subseteq \varphi(a)\}\right) = \operatorname{cl}(\operatorname{core}(a)).$$

(2) follows from (1). □

We now turn to morphisms between algebraic L-spaces.

Definition 4.6.
(1) We call an L-morphism $f\colon X_1 \to X_2$ between L-spaces *coherent* if
$$f^{-1}(\operatorname{core} U) \subseteq \operatorname{core} f^{-1}(U) \qquad \text{for all } U \in \mathrm{ClopUp}(X_2).$$
(2) Let AlgLPries be the category of algebraic L-spaces and coherent L-morphisms.

It is easy to see that the identity morphism is a coherent L-morphism and that the composition of two coherent L-morphisms is coherent. Therefore, AlgLPries is indeed a category. We show that AlgLPries is a full subcategory of ConLPries. For this we need the following lemmas.

Lemma 4.7 ([12, Lemma 4.14(1)]). *Let X be an L-space and U an open upset of X. Then* $\operatorname{cl} U \cap Y = U \cap Y$.

Lemma 4.8. *Let X be a continuous L-space and $U \in \mathrm{ClopUp}(X)$. The following are equivalent.*
(1) $\ker U = \operatorname{core} U$.
(2) $\operatorname{core} U$ *is dense in* U.
(3) *For each* $y \in U \cap Y$*, there is* $V \in \mathrm{ClopSUp}(X)$ *such that* $y \in V \subseteq U$.
(4) *For each Scott upset* $F \subseteq \ker U$*, there is* $V \in \mathrm{ClopSUp}(X)$ *such that* $F \subseteq V \subseteq U$.

Proof. (1)⇒(2) Since X is a continuous L-space, $\ker U$ is dense in U. Therefore, $\ker U = \operatorname{core} U$ implies that $\operatorname{core} U$ is dense in U.

(2)⇒(3) Suppose that $y \in U \cap Y$. Because $U = \operatorname{cl}(\operatorname{core} U)$, we have $y \in \operatorname{cl}(\operatorname{core} U) \cap Y$. Since $\operatorname{core} U$ is an open upset, $\operatorname{cl}(\operatorname{core} U) \cap Y = \operatorname{core} U \cap Y$ by Lemma 4.7. Therefore, $y \in \operatorname{core} U$, and so there is $V \in \mathrm{ClopSUp}(X)$ such that $y \in V \subseteq U$.

(3)⇒(4) Let $F \subseteq \ker U$ be a Scott upset and $y \in F \cap Y$. Then $y \in \ker U$, so $y \in U$ by Lemma 4.3(1). Therefore, by (3), there is $V_y \in \mathrm{ClopSUp}(X)$ such that $y \in V_y \subseteq U$. Thus,
$$F = \bigcup\{\uparrow y \mid y \in F \cap Y\} \subseteq \bigcup\{V_y \mid y \in F \cap Y\} \subseteq U.$$
Because F is closed, it is compact. Therefore, since a finite union of clopen Scott upsets is a clopen Scott upset, there is $V \in \mathrm{ClopSUp}(X)$ such that $F \subseteq V \subseteq U$.

(4)⇒(1) By Lemma 4.3(1), $\operatorname{core} U \subseteq \ker U$. For the reverse inclusion, it suffices to show that $V \ll U$ implies there is $W \in \mathrm{ClopSUp}(X)$ such that $V \subseteq W \subseteq U$. Let $V \ll U$. Then there is a Scott upset F such that $V \subseteq F \subseteq U$ (see, e.g., [12, Lemma 5.7]). But $U = \operatorname{cl}(\ker U)$, so $F \subseteq \ker U$ by Lemma 4.1. Therefore, by (4), there is $W \in \mathrm{ClopSUp}(X)$ such that $F \subseteq W \subseteq U$, and hence $V \subseteq W \subseteq U$. □

Lemma 4.9. *Let $f\colon X_1 \to X_2$ be an L-morphism between L-spaces.*
(1) *If f is proper and X_1 is an algebraic L-space, then f is coherent.*
(2) *If f is coherent and X_2 is an algebraic L-space, then f is proper.*

(3) *If X_1 and X_2 are algebraic L-spaces, then f is coherent iff f is proper.*

Proof. (1) Let $U \in \mathrm{ClopUp}(X_2)$. Then

$$\begin{aligned} f^{-1}(\operatorname{core} U) &\subseteq f^{-1}(\ker U) && \text{by Lemma 4.3(1)} \\ &\subseteq \ker f^{-1}(U) && \text{since } f \text{ is proper} \\ &= \operatorname{core} f^{-1}(U) && \text{by Lemmas 4.3(3) and 4.8(1).} \end{aligned}$$

(2) Let $U \in \mathrm{ClopUp}(X_2)$. Then

$$\begin{aligned} f^{-1}(\ker U) &= f^{-1}(\operatorname{core} U) && \text{by Lemmas 4.3(3) and 4.8(1)} \\ &\subseteq \operatorname{core} f^{-1}(U) && \text{since } f \text{ is coherent} \\ &\subseteq \ker f^{-1}(U) && \text{by Lemma 4.3(1).} \end{aligned}$$

(3) follows from (1) and (2). □

Putting Lemmas 4.3(3) and 4.9(3) together, we obtain the following:

Theorem 4.10. AlgLPries *is a full subcategory of* ConLPries.

We are ready to prove the first main result of this section.

Theorem 4.11. AlgFrm *is dually equivalent to* AlgLPries.

Proof. By Remark 2.2, AlgFrm is a full subcategory of ConFrm. By Theorem 4.10, AlgLPries is a full subcategory of ConLPries. Thus, the result follows from Theorems 3.10 and 4.5(2).

Finally, we connect AlgLPries with KBSob.

Lemma 4.12. *Let X be an SL-space, Y its spatial part, and $U \subseteq X$. Then $U \in \mathrm{ClopSUp}(X)$ iff there is a compact open set V of Y such that $\operatorname{cl} V = U$.*

Proof. By [11, Theorem 5.7], the poset of Scott upsets of X is isomorphic to the poset of compact saturated sets of Y. The isomorphism is obtained by sending a Scott upset $F \subseteq X$ to the compact saturated set $F \cap Y$, and a compact saturated set $K \subseteq Y$ to the Scott upset ${\uparrow}K$.

($\Rightarrow$) Suppose that U is a clopen Scott upset. Then $V := U \cap Y$ is a compact saturated subset of Y. Moreover, V is an open subset of Y since $U \in \mathrm{ClopUp}(X)$. Furthermore, $\operatorname{cl} V = U$ by Remark 3.6(2) because X is an SL-space.

($\Leftarrow$) Suppose there is a compact open set V of Y such that $\operatorname{cl} V = U$. Then ${\uparrow}V$ is a Scott upset of X. Since V is open and X is an SL-space, there is $U' \in \mathrm{ClopUp}(X)$ such that $V = U' \cap Y$ and $\operatorname{cl} V = U'$ (see Remark 3.6(2)). Therefore, $U = \operatorname{cl} V = U'$, and so U is a clopen upset of X. Moreover,

$$U = {\uparrow}U = {\uparrow}\operatorname{cl} V = \operatorname{cl}{\uparrow}V = {\uparrow}V,$$

where the third equality follows from [19, Theorem 3.1.2] since X is an Esakia space. Thus, U is a Scott upset. □

Theorem 4.13. *Let X be an SL-space and Y its spatial part. Then X is an algebraic L-space iff Y is a compactly based sober space.*

Proof. Since the spatial part of an L-space is always sober (see, e.g., [12, Lemma 4.11]), it is sufficient to show that X is an algebraic L-space iff Y is compactly based. First suppose that X is an algebraic L-space. Let $V \subseteq Y$ be open and $y \in V$. Set $U = \operatorname{cl} V$. Then U is a clopen upset of X by Remark 3.6(2). Moreover, it follows from [12, Lemma 4.14(2)] that

$$U \cap Y = \operatorname{cl} V \cap Y = V,$$

so $y \in U \cap Y$. By Lemmas 4.3(3) and 4.8(3), there is $W \in \operatorname{ClopSUp}(X)$ such that $y \in W \subseteq U$. Therefore, $y \in W \cap Y \subseteq U \cap Y = V$. It follows from the proof of Lemma 4.12 that $W \cap Y$ is a compact open subset of Y. Thus, Y is compactly based.

Conversely, suppose that Y is compactly based and $U \in \operatorname{ClopUp}(X)$. Since Y is locally compact, X is a continuous L-space by Theorem 3.10. Therefore, by Lemma 4.8(3), it suffices to show that for each $y \in U \cap Y$ there is $V \in \operatorname{ClopSUp}(X)$ such that $y \in V \subseteq U$. Because $U \cap Y$ is an open subset of Y and Y is compactly based, there is a compact open $K \subseteq Y$ such that $y \in K \subseteq U \cap Y$. Therefore, $\operatorname{cl} K \in \operatorname{ClopSUp}(X)$ by Lemma 4.12. Moreover, $y \in \operatorname{cl} K \subseteq \operatorname{cl}(U \cap Y) = U$, where in the last equality we use that X is an SL-space. Thus, X is an algebraic L-space. □

By Theorem 4.10, AlgLPries is a full subcategory of ConLPries. By Remark 2.9, KBSob is a full subcategory of LKSob. Thus, as an immediate consequence of Theorems 3.10 and 4.13, we obtain the following:

Corollary 4.14. AlgLPries *is equivalent to* KBSob.

Putting together Theorem 4.11 and Corollary 4.14, we obtain Theorem 2.12(1) that AlgFrm is dually equivalent to KBSob.

5. Priestley duality for arithmetic, coherent, and Stone frames

In this final section we describe Priestley duals of arithmetic, coherent, and Stone frames. We also connect them to stably compactly based, spectral, and Stone spaces, thus obtaining an alternative proof of Theorem 2.12(2,3,4). We conclude the paper by pointing out a connection to Priestley duality for bounded distributive lattices and Stone duality for boolean algebras.

5.1. Arithmetic frames

We recall (see Definition 3.11(1a)) that an L-space X is kernel-stable provided $\ker(U \cap V) = \ker U \cap \ker V$ for all $U, V \in \operatorname{ClopUp}(X)$.

Definition 5.1.

(1) An *arithmetic L-space* is a kernel-stable algebraic L-space.
(2) Let AriLPries be the full subcategory of AlgLPries consisting of arithmetic L-spaces.

Lemma 5.2. *Let X be an algebraic L-space. Then X is an arithmetic L-space iff $U_1 \cap U_2 \in \operatorname{ClopSUp}(X)$ for every $U_1, U_2 \in \operatorname{ClopSUp}(X)$.*

Proof. For the left-to-right implication, let $U_1, U_2 \in \mathrm{ClopSUp}(X)$. It follows from Lemma 4.4 that $\ker U_1 = U_1$ and $\ker U_2 = U_2$. Therefore, since X is kernel-stable,

$$\ker(U_1 \cap U_2) = \ker U_1 \cap \ker U_2 = U_1 \cap U_2.$$

Thus, $U_1 \cap U_2 \in \mathrm{ClopSUp}(X)$ using Lemma 4.4 again.

For the right-to-left implication, let $U_1, U_2 \in \mathrm{ClopUp}(X)$. It suffices to show that for each $W \in \mathrm{ClopUp}(X)$ we have

$$W \subseteq \ker U_1 \cap \ker U_2 \iff W \subseteq \ker(U_1 \cap U_2).$$

Since W is compact, by the assumption that $V_1, V_2 \in \mathrm{ClopSUp}(X)$ implies $V_1 \cap V_2 \in \mathrm{ClopSUp}(X)$ and Lemma 4.8,

$$\begin{aligned} W \subseteq \ker U_1 \cap \ker U_2 &\iff W \subseteq \operatorname{core} U_1 \cap \operatorname{core} U_2 \\ &\iff \exists V_1, V_2 \in \mathrm{ClopSUp}(X) : W \subseteq V_1 \subseteq U_1 \text{ and } \\ &\qquad\qquad\qquad\qquad\qquad\qquad W \subseteq V_2 \subseteq U_2 \\ &\iff \exists V \in \mathrm{ClopSUp}(X) : W \subseteq V \subseteq U_1 \cap U_2 \\ &\iff W \subseteq \operatorname{core}(U_1 \cap U_2) \\ &\iff W \subseteq \ker(U_1 \cap U_2). \end{aligned}$$

□

Lemma 5.3. *Let Y be a compactly based sober space. Then Y is stably locally compact iff Y is stably compactly based.*

Proof. The left-to-right implication is trivial. For the right-to-left implication, let $A, B \subseteq Y$ be compact saturated. Since Y is compactly based, every compact saturated set is an intersection of compact open sets (see Remark 2.9). Therefore, $A \cap B = \bigcap \mathcal{F}$, where

$$\mathcal{F} = \{U \cap V \mid U, V \text{ compact open with } A \subseteq U \text{ and } B \subseteq V\}.$$

Since Y is stably compactly based, $\mathcal{F}$ is closed under finite intersections. Thus, the Hofmann-Mislove Theorem (see, e.g., [20, Corollary II−1.22]) implies that $\bigcap \mathcal{F}$ is compact. Consequently, $A \cap B$ is compact. □

Theorem 5.4. *Let L be an algebraic frame, X_L its Priestley space, and Y_L the spatial part of X_L. The following are equivalent.*

1. *L is an arithmetic frame.*
2. *X_L is an arithmetic L-space.*
3. *Y_L is a stably compactly based space.*

Proof. Since L is an algebraic frame, X_L is an algebraic L-space by Theorem 4.5(2), and hence Y_L is a compactly based sober space by Theorem 4.13.

(1)⇔(2) Let L be an arithmetic frame and $\varphi(a), \varphi(b) \in \mathrm{ClopSUp}(X_L)$. Then $a, b \in K(L)$ by Lemma 4.4. Since L is an arithmetic frame, $a \wedge b \in K(L)$. Therefore, $\varphi(a) \cap \varphi(b) = \varphi(a \wedge b)$ is a Scott upset, again by Lemma 4.4. Thus, X_L is an arithmetic L-space by Lemma 5.2.

Conversely, let X_L be an arithmetic L-space and $a, b \in K(L)$. By Lemma 4.4, $\varphi(a)$ and $\varphi(b)$ are clopen Scott upsets. Therefore, $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$ is a Scott upset by Lemma 5.2. Thus, $a \wedge b \in K(L)$, again by Lemma 4.4. Hence, L is an arithmetic frame.

(2)⇔(3) Since X_L is an algebraic L-space, X_L is an arithmetic L-space iff X_L is a stably continuous L-space by Lemma 5.2. But X_L is a stably continuous L-space iff Y_L is a stably locally compact space by [12, Theorem 6.7]. However, since Y_L is a compactly based sober space, Y_L is stably locally compact iff Y_L is stably compactly based by Lemma 5.3. Thus, X_L is an arithmetic L-space iff Y_L is a stably compactly based space. □

As a consequence of Theorem 4.11, Corollary 4.14, and Theorem 5.4, we arrive at the first main result of this section:

Theorem 5.5. *The category* AriLPries *is equivalent to* StKBSp *and dually equivalent to* AriFrm*.*

As a corollary we obtain Theorem 2.12(2), which states that AriFrm is dually equivalent to StKBSp.

5.2. Coherent frames

We next turn our attention to Priestley duals of coherent frames. Since coherent frames are exactly compact arithmetic frames, we obtain that Priestley duals of coherent frames are exactly arithmetic L-spaces that are L-compact (see Lemma 4.4). We then connect L-compact arithmetic L-spaces with spectral spaces to obtain the well-known duality between CohFrm and Spec.

Definition 5.6.

(1) A *coherent L-space* is an L-compact arithmetic L-space.
(2) Let CohLPries be the full subcategory of AriLPries consisting of coherent L-spaces.

Lemma 5.7 ([12, Lemma 6.15]). *Let X be an SL-space and Y its spatial part. Then X is L-compact iff Y is compact.*

Theorem 5.8. *Let L be an algebraic frame, X_L its Priestley space, and Y_L the spatial part of X_L. The following are equivalent.*

(1) *L is a coherent frame.*
(2) *X_L is a coherent L-space.*
(3) *Y_L is a spectral space.*

Proof. (1)⇔(2) L is a coherent frame iff L is a compact arithmetic frame. By Lemma 4.4 and Theorem 5.4, this is equivalent to X_L being a coherent L-space.

(2)⇔(3) By Lemma 5.7 and Theorem 5.4, X_L is a coherent L-space iff Y_L is a compact stably compactly based space, hence a spectral space. □

As a consequence of Theorems 5.5 and 5.8, we obtain the second main result of this section:

Corollary 5.9. *The category* CohLPries *is equivalent to* Spec *and dually equivalent to* CohFrm*.*

As a corollary we obtain Theorem 2.12(3), which states that CohFrm is dually equivalent to Spec.

5.3. Stone frames

Finally, we describe Priestley duals of Stone frames. Stone frames are characterized by having enough complemented elements. In the language of Priestley spaces, complemented elements correspond to clopen upsets that are also downsets (see, e.g., [9, Lemma 6.1]).

Let X be a Priestley space. Following [9, p. 377], we call a subset of X a *biset* if it is both an upset and a downset. Let $\mathrm{ClopBi}(X)$ be the collection of clopen bisets of X.

Definition 5.10. Let X be an L-space.

(1) For $U \in \mathrm{ClopUp}(X)$, define the *center of* U as

$$\mathrm{cen}\, U = \bigcup \{V \in \mathrm{ClopBi}(X) \mid V \subseteq U\}.$$

(2) We call X a *zero-dimensional L-space* if $\mathrm{cen}\, U$ is dense in U for every $U \in \mathrm{ClopUp}(X)$.

(3) A *Stone L-space* is a zero-dimensional L-space that is L-compact.

(4) Let $\mathsf{StoneLPries}$ be the full subcategory of LPries consisting of Stone L-spaces.

Remark 5.11. In [9, Definition 6.2], the center of a clopen upset U is called the biregular part of U.

Lemma 5.12. *Let X be an L-space and $U \in \mathrm{ClopUp}(X)$.*

(1) $\mathrm{cen}\, U \subseteq \mathrm{reg}\, U$.

(2) *If X is a zero-dimensional L-space, then X is a regular L-space.*

(3) *If X is a Stone L-space, then X is a compact regular L-space.*

Proof. (1) Suppose $x \in \mathrm{cen}\, U$. Then there is $V \in \mathrm{ClopBi}(X)$ with $x \in V \subseteq U$. Therefore, ${\downarrow}{\uparrow}x \subseteq U$, so $x \in \mathrm{reg}\, U$ by [12, Lemma 7.3(1)].

(2) Suppose $U \in \mathrm{ClopUp}(X)$. Since X is a zero-dimensional L-space, $\mathrm{cen}\, U$ is dense in U. But then $\mathrm{reg}\, U$ is dense in U by (1). Thus, X is a regular L-space.

(3) This follows from (2). □

As an immediate consequence, we obtain that $\mathsf{StoneLPries}$ is a full subcategory of $\mathsf{KRLPries}$. We proceed to show that $\mathsf{StoneLPries}$ is a full subcategory of $\mathsf{CohLPries}$.

Lemma 5.13. *Let X be a Stone L-space.*

(1) $\mathrm{ClopSUp}(X) = \mathrm{ClopBi}(X)$.

(2) $\mathrm{cen}\, U = \mathrm{reg}\, U = \mathrm{core}\, U$ *for each* $U \in \mathrm{ClopUp}(X)$.

Proof. (1) Since X is a Stone L-space, it is a compact regular L-space by Lemma 5.12(3). Therefore, Scott upsets are exactly closed bisets by [12, Lemma 7.15(4)], and the result follows.

(2) That $\mathrm{cen}\, U \subseteq \mathrm{reg}\, U$ follows from Lemma 5.12(1). We show that $\mathrm{reg}\, U \subseteq \mathrm{core}\, U$. Let $x \in \mathrm{reg}\, U$. Then there is $V \in \mathrm{ClopUp}(X)$ such that $x \in V$ and ${\downarrow}V \subseteq U$. Hence, ${\downarrow}x \subseteq U$, and therefore ${\uparrow}{\downarrow}x \subseteq U$. But since X

is L-compact, $\min(\downarrow x) \subseteq \min X \subseteq Y$ by Lemma 4.4, and so $\uparrow\downarrow x$ is a Scott upset by Lemma 4.1. Because X is an algebraic L-space, $\operatorname{cl}\operatorname{core} U = U$. Thus, $\uparrow\downarrow x \subseteq \operatorname{core} U$ by Lemma 4.1. Finally, we show that $\operatorname{core} U = \operatorname{cen} U$. For this it suffices to show that for each clopen upset V we have $V \subseteq \operatorname{cen} U$ iff $V \subseteq \operatorname{core} U$. Since V is compact, finite unions of bisets are bisets, and finite unions of Scott upsets are Scott upsets, (1) implies

$$\begin{aligned} V \subseteq \operatorname{cen} U &\iff \exists W \in \operatorname{ClopBi}(X) : V \subseteq W \subseteq U \\ &\iff \exists W \in \operatorname{ClopSUp}(X) : V \subseteq W \subseteq U \\ &\iff V \subseteq \operatorname{core} U. \end{aligned}$$ □

Theorem 5.14. StoneLPries *is a full subcategory of* CohLPries.

Proof. Every Stone L-space is a coherent L-space by Lemma 5.13(2). Also, since StoneLPries is a full subcategory of KRLPries, every L-morphism between Stone L-spaces is a proper L-morphism by [12, Theorem 7.18(2)]. Therefore, every such morphism is a coherent L-morphism by Lemma 4.9(3). Thus, StoneLPries is a full subcategory of CohLPries. □

In [9, Theorem 6.3(1)] it is shown that Priestley duals of zero-dimensional frames are exactly zero-dimensional L-spaces. We connect zero-dimensional L-spaces to zero-dimensional topological spaces.

Lemma 5.15. *Let X be an L-space and Y its spatial part.*
(1) *If $U \in \operatorname{ClopBi}(X)$, then $U \cap Y$ is clopen in Y.*
(2) *If X is an SL-space and $V \subseteq Y$ is clopen, then there is $U \in \operatorname{ClopBi}(X)$ such that $V = U \cap Y$.*

Proof. (1) This is immediate.

(2) Let $V \subseteq Y$ be clopen. Since V is open, there is $U \in \operatorname{ClopUp}(X)$ such that $V = U \cap Y$ and $\operatorname{cl} V = U$ (see Remark 3.6(2)). Similarly, because V is closed, there is $W \in \operatorname{ClopUp}(X)$ such that $Y \backslash V = W \cap Y$ and $\operatorname{cl}(Y \backslash V) = W$. Since $V, Y \backslash V$ are open in Y, we have $\operatorname{cl}(V) \cap \operatorname{cl}(Y \setminus V) = \operatorname{cl}(V \cap (Y \setminus V))$ by [12, Lemma 4.15]. Therefore, $U \cap W = \operatorname{cl}(V) \cap \operatorname{cl}(Y \backslash V) = \varnothing$. Also,

$$U \cup W = \operatorname{cl} V \cup \operatorname{cl}(Y \backslash V) = \operatorname{cl}(V \cup (Y \backslash V)) = \operatorname{cl} Y = X.$$

Thus, $U = X \backslash W$, and hence $U \in \operatorname{ClopBi}(X)$. □

Theorem 5.16. *Let X be an L-space and Y its spatial part.*
(1) *If X is a zero-dimensional L-space, then Y is zero-dimensional.*
(2) *If X is an SL-space, then X is a zero-dimensional L-space iff Y is zero-dimensional.*

Proof. (1) Suppose X is a zero-dimensional L-space. Let $V \subseteq Y$ be open and $y \in V$. Then there is $U \in \operatorname{ClopUp}(X)$ such that $U \cap Y = V$. Since $\operatorname{cen} U$ is dense in U, we have $U \cap Y = \operatorname{cl}(\operatorname{cen} U) \cap Y = \operatorname{cen} U \cap Y$, where the last equality follows from Lemma 4.7 because $\operatorname{cen} U$ is an open upset of X. Therefore, there is $W \in \operatorname{ClopBi}(X)$ such that $y \in W \subseteq U$. Thus, $y \in W \cap Y \subseteq V$ and $W \cap Y$ is clopen in Y by Lemma 5.15(1). Consequently, Y is zero-dimensional.

(2) The left-to-right implication follows from (1). For the converse, suppose Y is zero-dimensional. Let $U \in \mathrm{ClopUp}(X)$. Since X is an SL-space, $U \cap Y$ is dense in U. Therefore, it suffices to show that $U \cap Y \subseteq \mathrm{cen}\, U$. Let $y \in U \cap Y$. Since $U \in \mathrm{ClopUp}(X)$, we have that $U \cap Y$ is open in Y. Because Y is zero-dimensional, there is clopen $V \subseteq Y$ such that $y \in V \subseteq U \cap Y$. Since V is clopen in Y, Lemma 5.15(2) implies that there is $W \in \mathrm{ClopBi}(X)$ such that $V = W \cap Y$. Because X is an SL-space, $\mathrm{cl}\, V = W$, and hence $y \in W \subseteq U$. Thus, $y \in \mathrm{cen}\, U$. □

Corollary 5.17. *Let L be a frame, X_L its Priestley space, and Y_L the spatial part of X_L. The following are equivalent.*

(1) *L is a zero-dimensional frame.*
(2) *X_L is a zero-dimensional L-space.*

If in addition L is spatial, then (1) *and* (2) *are equivalent to*

(3) *Y_L is a zero-dimensional space.*

Proof. The equivalence (1)⇔(2) is shown in [9, Theorem 6.3(1)], and (2)⇔(3) follows from Theorem 5.16(2). □

Corollary 5.18. *Let L be a frame, X_L its Priestley space, and Y_L the spatial part of X_L. The following are equivalent.*

(1) *L is a Stone frame.*
(2) *X_L is a Stone L-space.*

If in addition L is spatial, then (1) *and* (2) *are equivalent to*

(3) *Y_L is a Stone space.*

Proof. (1)⇔(2) Apply Lemma 4.4 and Corollary 5.17.

(2)⇔(3) Apply Lemma 5.7 and Corollary 5.17. □

As an immediate consequence, we arrive at the last main result of this section:

Corollary 5.19. StoneLPries *is equivalent to* Stone *and dually equivalent to* StoneFrm.

Proof. This follows from Corollaries 5.9 and 5.18 and the observation that StoneFrm, StoneLPries, and Stone are full subcategories of CohFrm, CohLPries, and Spec, respectively (see Remark 2.7, Theorem 5.14 and Remark 2.9). □

Theorem 2.12(4), which states that StoneFrm is dually equivalent to Stone, is now immediate from the above corollary.

Remark 5.20. Let L be a frame and X_L its Priestley space. As we saw in this paper, there are various maps from the clopen upsets of X_L to the open upsets of X_L, and the corresponding density conditions are responsible for various properties of L. In particular,

- L is continuous iff $\ker U$ is dense in U for each $U \in \mathrm{ClopUp}(X)$;
- L is algebraic iff $\mathrm{core}\, U$ is dense in U for each $U \in \mathrm{ClopUp}(X)$;

$$
\begin{array}{ccccc}
\mathsf{AlgFrm} & \leftrightsquigarrow & \mathsf{AlgLPries} & \longleftrightarrow & \mathsf{KBSob} \\
\rotatebox[origin=c]{90}{$\subsetneq$} & & \rotatebox[origin=c]{90}{$\subsetneq$} & & \rotatebox[origin=c]{90}{$\subsetneq$} \\
\mathsf{AriFrm} & \leftrightsquigarrow & \mathsf{AriLPries} & \longleftrightarrow & \mathsf{StKBSp} \\
\rotatebox[origin=c]{90}{$\subsetneq$} & & \rotatebox[origin=c]{90}{$\subsetneq$} & & \rotatebox[origin=c]{90}{$\subsetneq$} \\
\mathsf{CohFrm} & \leftrightsquigarrow & \mathsf{CohLPries} & \longleftrightarrow & \mathsf{Spec} \\
\rotatebox[origin=c]{90}{$\subsetneq$} & & \rotatebox[origin=c]{90}{$\subsetneq$} & & \rotatebox[origin=c]{90}{$\subsetneq$} \\
\mathsf{StoneFrm} & \leftrightsquigarrow & \mathsf{StoneLPries} & \longleftrightarrow & \mathsf{Stone}
\end{array}
$$

FIGURE 6. Equivalences and dual equivalences between various categories of algebraic frames, algebraic L-spaces, and compactly based sober spaces

- L is regular iff $\operatorname{reg} U$ is dense in U for each $U \in \operatorname{ClopUp}(X)$;
- L is zero-dimensional iff $\operatorname{cen} U$ is dense in U for each $U \in \operatorname{ClopUp}(X)$.

The strength of these properties of frames is then described by how these maps interact. For example, $\operatorname{core} U \subseteq \ker U$ for each $U \in \operatorname{ClopUp}(X)$ indicates that every algebraic frame is continuous, etc.

To summarize, we have the diagram in Figure 6, where we use the same notation as in the previous diagrams. An overview of the introduced categories of Priestley spaces is given in Table 6, where the numbers in parentheses indicate the corresponding definitions in the text. The relevant categories of frames and spaces are described in Tables 2 and 4.

We conclude the paper by connecting the results obtained above with Priestley duality for bounded distributive lattices and Stone duality for boolean algebras. Let D be a bounded distributive lattice, X_D its Priestley space, and $\varphi_D : D \to \operatorname{ClopUp}(X_D)$ the Stone map. We denote by π_D the topology of X_D and by τ_D the topology of open upsets of X_D. Then $\{\varphi_D(a) \mid a \in D\}$ is a basis for τ_D. Moreover, since $\{\varphi_D(a) \setminus \varphi_D(b) \mid a, b \in D\}$ is a basis for π_D, we see that π_D is the patch topology of τ_D.

Let DLat be the category of bounded distributive lattices and bounded lattice homomorphisms. By the well-known equivalence between DLat and CohFrm (see, e.g., [29, p. 65]), each bounded distributive lattice D is isomorphic to the lattice $K(L)$ of compact elements of a coherent frame L. Let X_D be the Priestley space of D, X_L the Priestley space of L, and Y_L the spatial part of X_L. Identifying D with $K(L)$, the map $P \mapsto P \cap D$ is an isomorphism between

TABLE 6. Categories of algebraic L-spaces

Category	Objects	Morphisms
$\mathsf{AlgLPries}$	Algebraic L-spaces (4.2)	Coherent L-morphisms (4.6)
$\mathsf{AriLPries}$	Arithmetic L-spaces (5.1)	Coherent L-morphisms
$\mathsf{CohLPries}$	Coherent L-spaces (5.6)	Coherent L-morphisms
$\mathsf{StoneLPries}$	Stone L-spaces (5.10)	L-morphisms (3.1)

$(Y_L, \subseteq)$ and $(X_D, \subseteq)$. However, π_D is different from the subspace topology on Y_L induced by π_L. Indeed, π_D is the patch topology of τ_D. By identifying X_D with Y_L, we have $\varphi_D(a) = \varphi_L(a) \cap Y_L$ for each $a \in D$. Since $\mathrm{ClopSUp}(X_L)$ corresponds to D (see Lemma 4.4), π_D is generated by the basis

$$\{(U \setminus V) \cap Y_L \mid U, V \in \mathrm{ClopSUp}(X_L)\}.$$

Thus, π_D is the patch topology of the subspace topology on Y_L induced by τ_L. We next show that this topology may not be the subspace topology induced by π_L.

Let D be a boolean algebra. Then X_D is a Stone space and L is a Stone frame. In this case, $\pi_D = \tau_D$, and hence X_D is realized as Y_L with the subspace topology induced by τ_L. On the other hand, since each Stone frame is a compact Hausdorff frame, $Y_L = \min X_L$ (see, e.g., [12, Lemma 7.15(5)]). Thus, Y_L is exactly the set of isolated points of X_L, and so the subspace topology on Y_L induced by π_L is discrete. This shows that the restrictions of τ_L and π_L to Y_L are distinct, yielding that the operations of taking the patch topology and the subspace topology may not commute.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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